

04-19-10

Jens Funke - Spectacle

Cycles and Modular Forms of Half-integral weight

(I)

Motivation: Let $G = \text{So}(p, q)^\circ$, ^{identity component}

$\mathcal{K} = \text{So}(p) \times \text{So}(q)$. G/\mathcal{K} is a symmetric space. Let P = arithmetic lattice in G .

Then if $D = G/P$, the P is

a locally symmetric space (real manifold with dimension pq)

Ex: $O(2, 1)$, then $D \cong \mathbb{H} = \text{upper half plane}$.

Reason: let V = quadratic space with signature $(2, 1)$.

~~V~~ $V = \{X \in M_2(\mathbb{C}) : \text{tr}(X) = 0\}$.

$\sqrt{(X, X)} = -\det(X)$. SL_2 acts on V by conjugation.

$\rightsquigarrow PSL(2, \mathbb{R}) \cong \text{So}(2, 1)^\circ$.

Then, since if G is $SL(2, \mathbb{R})$ we get

the D for $SL(2, \mathbb{R})$ is \mathbb{H} , we're done.

(1)

Question: 1) $H^*(X) \cong H^*(\Gamma) ??$

2) Geometric interpretation of these cohomology classes? ~~Repn's~~ (As Poincaré duals of nice submanifolds).

3) What is $H^*(X, \mathbb{E})$ and $H^*(X, E)$, where

~~pointwise~~ E is a \mathbb{R}^n dim'l rep'n of Γ ?

(II)

Case of $O(2, 1)$. So $G \cong PSL(2, \mathbb{R})$.

$V =$ quadratic space $= \{X \in M_2(\mathbb{Q}) \text{ s.t. } \text{tr}(X) = 0\}$

$\Gamma \subset SL_2(\mathbb{Z})$ congruence subgroup

such that Γ stabilizes a lattice \mathbb{L} in V .

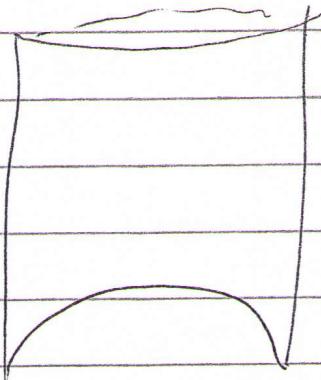
$X = \frac{\mathbb{H}}{\Gamma}$ = modular curve.

Let $\bar{X} =$ Borel-Serre compactification of

X , which is a manifold with boundary.

(2)

Ex: For $SL_2(\mathbb{Z})$, the \overline{X} is



i.e. you don't add a point at ∞ ,
but ~~but~~ you add a circle at the top.

Then, $H^*(X) \cong H^*(\overline{X})$.

Let $E = E_{2k} = \text{Sym}^k(\mathbb{Q}^2)$ be the

symmetric power of the standard repn. It
has highest weight $2k$.

Then $E \cong \mathcal{H}^k(V) = \left\{ \begin{array}{l} \text{harmonic } k\text{-tensors} \\ \text{ie. killed by the Laplacian} \end{array} \right\}$

A

$\text{Sym}^k(V)$

(3)

Eichler-Shimura gives us a way to construct cohomology classes. ~~It does~~

Eichler-Shimura say if

$$f \in M_{2k+2}(\Gamma)$$

modular form of weight $2k$

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

then ~~γ_f~~ $\gamma_f := f(z)dz \otimes \chi(z) V_{-2k}$

$$\hookrightarrow A^{\text{top}}(X, \tilde{E}_{-2k})$$

Then $H^1(X, \tilde{E}_{2k}) \cong M_{2k+2}(\Gamma) \oplus S_{2k+2}(\Gamma)$

cusp forms

$$S_{2k+2}(\Gamma) \subset H^1_c(X, \tilde{E}_{2k})$$

~~III~~

Modular Symbols / Special Cycles

let $x = \begin{pmatrix} b & ac \\ -2a & -b \end{pmatrix} \in V$, $(x, x) > 0$.

Then consider $D_x := \left\{ z \in D \cong \mathbb{H} : \begin{array}{l} |z|^2 + bR_E(z) \\ + c = 0 \end{array} \right\}$

D_x is a geodesic in \mathbb{H} .

If $x = \begin{pmatrix} b \\ -b \end{pmatrix}$, then

$D_x = \text{imaginary axis}$.

Let Γ_x be the stabilizer of x in Γ .

~~at~~ $\Gamma_x \subset D_x$. let $C_x := D_x / \Gamma_x$

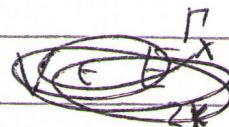
$C_x = \begin{cases} \text{closed geodesic if } \overline{\Gamma_x} \neq 1 \\ \text{infinite if } \overline{\Gamma_x} = 1 \end{cases}$

(5)

$$\text{Let } C_h := \sum_{\substack{x \in L \\ (x, x) = h \gg \\ \text{mod } p}} c_x \in H_1(\bar{X}, \partial \bar{X}, \mathbb{Z})$$

for $h \gg$.

Coefficient systems: If $v \in \mathbb{E}_{2k}^{\Gamma_X}$,

Γ_X -invariant


then the pair (c_x, v) defines naturally an

element in $H_1(X, \partial X, \mathbb{E}_{2k}^{\Gamma_X})$, since have ^{maps}  $\rightarrow H_1(X, \partial X, \tilde{\mathbb{E}}_{2k})$

Now, $x^k \in \mathbb{J}_m^k(v) \overset{\Gamma_X}{\circledast}$, so we can get a cycle.

Project ~~project~~ x^k to $\mathcal{H}^k(V) \overset{\Gamma_X}{\circledast}$, and

 call this result v .

So get a cycle (c_x, v) . Call this cycle $c_{x,k}$

 this

Prop.: Let $x = \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix} \in V$, so

$$c_x = \frac{1}{\sqrt{-1}}$$

Let $f \in S_{2k+2}(\Gamma)$, so get γ_f .

Then $\langle \gamma_f, c_{x,k} \rangle = * L(f, k+1)$

Completed L-function of f

evaluated at $k+1$.

IV

Shintani:

Thm: Let $f \in S_{2k+2}(\Gamma)$. Define

$$\rho(\tau, f) := \sum_{h>0} \langle \gamma_f, c_{h,k} \rangle e^{2\pi i h \tau}$$

Then $\rho(\tau, f) \in S_{k+3/2}(\Gamma)$

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IV

Spectacle Cycles: Let $K > 0$.

$$x = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad c_{x,k} \in H_1(\bar{X}, \partial\bar{X}, -)$$

$$\delta c_{x,k} = i\infty \otimes v_x - 0 \otimes v_x \in H_0(X_\infty, \widetilde{\mathbb{E}}_{2k})$$

where X_∞ is the circle we put on top of

$\begin{array}{c} H \\ \downarrow \\ SL(2, \mathbb{Z}) \end{array}$ toibel-Serre compactify it.

Prop: ($K > 0$) The 0-cycle $i\infty \otimes v_x$ is trivial.

\exists 1-chain in X_∞ with coefficients
whose boundary is $i\infty \otimes v_x$.

Millson-Furukawa

Thm (\downarrow): ~~The same~~, ~~theorem~~, ~~statement~~ proposition from previous page
~~Assume~~, holds, for ~~the~~ replacing
 $\mathbb{E}_{2k+2}(P)$ by $M_{2k+2}(P)$ and $c_{x,k}$ by
 $c_{X,k}$, which I didn't write the

definition here since I ~~didn't~~ ~~had~~
just didn't have the time. ~~They~~ ~~are~~ ~~not~~ ~~relevant~~
~~closed~~

Thm (M-F): ~~So~~ Ramanujan theorem holds when

you replace $C_{x,k}$ by $C_{x,k}^c$ and

$S_{2k+2}(r)$ by $M_{2k+2}(r)$.